

An analogue of the Magnus problem for associative algebras

V. Dotsenko, N. Iyudu and D. Korytin

Abstract

We prove an analogue of the Magnus theorem for associative algebras without unity over arbitrary fields. Namely, if an algebra is given by $n + k$ generators and k relations and has an n -element system of generators, then this algebra is a free algebra of rank n .

Let us recall two group-theoretic statements concerning free subgroups in finitely presented groups. The Magnus theorem [1, 2] says that if a group G is given by $n + k$ generators and k relations and has a system of n generators, then G is a free group of rank n . The following theorem was proved by Romanowskii [3]: let G be a group given by $n + k$ generators and k relations. Then there exists a free subgroup of G of rank n . Moreover, generators of this free subgroup can be chosen from the initial generators of the group. Analogues of both theorems for semigroups were proved by Shneerson [4]. The second theorem (Lyndon condition) for associative algebras over finite fields was announced in [5] and proved in [6]. We present here the proof of the analog of Magnus theorem for associative algebras without unity over arbitrary field.

Main Theorem. *Let A be an associative algebra without unity given by $n + k$ generators and k relations over a field \mathbf{k} . If A has an n -element system of generators g_1, \dots, g_n , then A is a free algebra of rank n .*

Let us fix the presentation of A :

$$A = \mathbf{k}\langle X \rangle / I, \quad I = \text{id}(h_1, \dots, h_k), \quad h_i \in \mathbf{k}\langle X \rangle,$$

where $X = \{x_1, \dots, x_{n+k}\}$. We assume that the free associative algebra $\mathbf{k}\langle X \rangle$ is supplied by the ordinary degree function, with all variables having degree 1. Denote the linear part (homogeneous part of degree 1) of $g \in \mathbf{k}\langle X \rangle$ by Lg . Let us mention two easy general facts.

Proposition 1. *Let $V = \{v_1, \dots, v_n\}$ be a system of elements $V \subset \mathbf{k}\langle X \rangle$ and $W = \{w_1, \dots, w_n\}$ obtained from V by a non-degenerate linear transformation. Then algebraic independence of V is equivalent to algebraic independence of W .*

Corollary 2. *Let $V = \{v_1, \dots, v_n\}$, $V \subset \mathbf{k}\langle X \rangle$ be a system of elements of degree 1. Then V is algebraically independent if and only if V is linearly independent.*

Let g_1, \dots, g_n be a system of generators of A . For convenience, we denote h_j by g_{n+j} . We also denote linear parts of polynomials g_1, \dots, g_{n+k} as y_1, \dots, y_{n+k} : $Lg_j = y_j$.

Theorem 3. *The system $\{y_1, \dots, y_{n+k}\}$ is algebraically independent.*

Proof. According to Corollary 2, it suffices to verify that y_1, \dots, y_{n+k} are linearly independent. Since g_1, \dots, g_n generate the quotient $A = \mathbf{k}\langle x_1, \dots, x_{n+k} \rangle / \text{id}(g_{n+1}, \dots, g_{n+k})$, we have that

$$x_i = \Phi_i(g_1, \dots, g_n) + d_i, \quad d_i \in \text{id}(g_{n+1}, \dots, g_{n+k}) \quad (1)$$

for any $i = 1, \dots, n+k$. Comparing the linear parts in these relations, we obtain

$$x_i = \sum_{r=1}^{n+k} \alpha_r^i y_r, \quad \alpha_r^i \in \mathbf{k} \quad (2)$$

for any $i = 1, \dots, n+k$. Hence the $(n+k)$ -element system y_1, \dots, y_{n+k} is linearly independent. \square

Consider the isomorphism φ of free algebras $\varphi : \mathbf{k}\langle x_1, \dots, x_{n+k} \rangle \rightarrow \mathbf{k}\langle y_1, \dots, y_{n+k} \rangle$ defined by the formula $\varphi(x_i) = \sum_{r=1}^{n+k} \alpha_r^i y_r$.

Lemma 4. *The system g_1, \dots, g_n is algebraically independent in the algebra*

$$\mathbf{k}\langle x_1, \dots, x_{n+k} \rangle / \text{id}(g_{n+1}, \dots, g_{n+k})$$

if and only if the system $\varphi g_1, \dots, \varphi g_n$ is algebraically independent in the algebra

$$\mathbf{k}\langle y_1, \dots, y_{n+k} \rangle / \text{id}(\varphi g_{n+1}, \dots, \varphi g_{n+k}).$$

Proof. Since φ is an isomorphism of free associative algebras, the lemma follows. \square

Theorem 5. *The system $\varphi g_1, \dots, \varphi g_n$ is algebraically independent in the algebra*

$$A = \mathbf{k}\langle y_1, \dots, y_{n+k} \rangle / \text{id}(\varphi g_{n+1}, \dots, \varphi g_{n+k}).$$

Proof. By the described non-degenerate change of variables, we ensure that:

$$L\varphi g_i = y_i, \quad i = 1, \dots, n+k. \quad (3)$$

Indeed, since $\varphi(x_i)$ is equal to $\sum \alpha_k^i y_k$ and y_k are linearly independent, there exists an inverse transformation $y_j = \sum \beta_k^j x_k$. Hence

$$L\varphi g_j = \varphi Lg_j = \varphi y_j = \sum_k \beta_k^j \varphi x_k = \sum_{k,i} \beta_k^j \alpha_i^k \varphi y_i = y_j.$$

Suppose there exists $\Phi \in \mathbf{k}\langle z_1, \dots, z_n \rangle$, $\Phi \neq 0$ such that

$$\Phi(\varphi g_1, \dots, \varphi g_n) = d \quad \text{for } d \in \text{id}(\varphi g_{n+1}, \dots, \varphi g_{n+k}).$$

The property (3) allows us to compare the minimal homogeneous parts of $\Phi(\varphi g_1, \dots, \varphi g_n)$ and $d \in \text{id}(\varphi g_{n+1}, \dots, \varphi g_{n+k})$. Homogeneous part of minimal degree of the polynomial $\Phi(\varphi g_1, \dots, \varphi g_n)$ is $M(y_1, \dots, y_n) = \sum_{(i)} \gamma_{(i)} y_{i_1} \dots y_{i_l}$, where $M(z_1, \dots, z_n)$ is the homogeneous part of minimal degree of $\Phi(z_1, \dots, z_n)$. Indeed, $M(y_1, \dots, y_n) \neq 0$ because y_1, \dots, y_n are algebraically independent. Hence $\Phi(\varphi g_1, \dots, \varphi g_n)$ contains only variables y_1, \dots, y_n and does not contain y_{n+1}, \dots, y_{n+k} .

Now we shall prove a lemma, which shows that some minimal degree monomials of all non-zero elements of the ideal $\text{id}(\varphi g_{n+1}, \dots, \varphi g_{n+k})$ contain one of the variables y_{n+1}, \dots, y_{n+k} . This contradiction will show that $\Phi(z_1, \dots, z_n) = 0$ and therefore the elements $\varphi g_1, \dots, \varphi g_n$ are algebraically independent.

Let us consider the free associative algebra $\mathbf{k}\langle x_1, \dots, x_{n+k} \rangle$ with the ordering on the variables $x_1 < \dots < x_{n+k}$. It induces the degree-lexicographical ordering on the set of monomials $\langle X \rangle$: $x_{i_1} \dots x_{i_k} < x_{j_1} \dots x_{j_m}$ if $k < m$ or $k = m$ and there exists l such that $i_l < j_l$ and $i_s = j_s$ for $s < l$.

For $s \in \mathbf{k}\langle X \rangle$ by $m(s)$ we denote the minimal monomial (with respect to the above ordering) which appears in s with non-zero coefficient.

Lemma 6. *Let $f_1, \dots, f_k \in \mathbf{k}\langle x_1, \dots, x_{n+k} \rangle$ and the linear part of f_i is x_i for any i : $Lf_i = x_i$. Let also $I = \text{id}(f_1, \dots, f_k)$ be the ideal generated by f_1, \dots, f_k . Then for any $s \in I$, its minimal monomial $m(s)$ contains at least one of the variables x_1, \dots, x_k .*

Proof. For an arbitrary presentation

$$s = \Pi(s) = \sum_{i, \alpha_i} c_{\alpha_i} p_{\alpha_i} f_i q_{\alpha_i},$$

where $c_{\alpha_i} \in \mathbf{k} \setminus \{0\}$ and $p_{\alpha_i}, q_{\alpha_i} \in \langle X \rangle$, we define the *parameter* $\tau = \tau(\Pi(s))$ of the presentation $\Pi(s)$ as the minimal monomial appearing in the presentation:

$$\tau = \min_{i, \alpha_i, j} p_{\alpha_i} u_j(f_i) q_{\alpha_i},$$

where $u_j(f_i)$ are monomials appearing in the polynomial f_i . We shall prove that under our conditions for each $s \in \text{id}(f_1, \dots, f_k)$, there exists a presentation $\Pi(s)$ of s with the parameter $m(s)$. We start with an arbitrary presentation of s :

$$s = \sum_{\alpha_i, i=1, \dots, k} c_{\alpha_i} p_{\alpha_i} f_i q_{\alpha_i},$$

where $c_{\alpha_i} \in \mathbf{k} \setminus \{0\}$ and $p_{\alpha_i}, q_{\alpha_i}$ are monomials. Consider those elements $p_{\alpha_i} f_i q_{\alpha_i}$ of this sum for which $p_{\alpha_i} x_i q_{\alpha_i}$ equals to the parameter τ of this presentation. Clearly $\tau \leq m(s)$. If $\tau = m(s)$, we are through. Assume that $\tau < m(s)$ and consider

$$M = \sum_{i, \alpha_i: p_{\alpha_i} x_i q_{\alpha_i} = \tau} c_{\alpha_i} p_{\alpha_i} f_i q_{\alpha_i}.$$

Since $\tau < m(s)$, we have $M \neq 0$. We will obtain a new presentation of s if in $\Pi(s)$ we replace all x_i with $1 \leq i \leq k$ by $f_i - (f_i - x_i)$ in all monomials $p_{\alpha_i}, q_{\alpha_i}$ from the sum M . Since $M \neq 0$, the same sum with x_i replaced by f_i also equals zero. Moreover, if we eliminate the last sum from the new presentation, we again obtain a presentation of s whose parameter is greater than τ since $m(f_i - x_i) > x_i$. Hence we have obtained a presentation of s with a greater parameter. If the new parameter is still less than $m(s)$, we can apply the same procedure to get another presentation with greater parameter etc. Since it can not become greater than $m(s)$, the process will terminate and we will obtain a presentation of s with parameter $m(s)$. Since the parameter of any presentation via f_1, \dots, f_k contains at least one of the variables x_1, \dots, x_k , we obtain the statement of the lemma. \square

As it was already mentioned above, the proof of Lemma 6 completes the proof of Theorem 5. \square

Now the Main Theorem follows from Lemma 4 and Theorem 5.

Acknowledgements. We would like to thank Professor Viktor Latyshev, who draw our attention to this very nice combinatorial problem.

References

- [1] W. Magnus, *Über freie Faktorgruppen und freie Untergruppen gegebener Gruppen*, Monathsh. Math. Phys. **47** (1939), 307–313
- [2] W. Magnus, A. Karrass and D. Solitar *Combinatorial group theory: Presentation of groups in terms of generators and relations*, Wiley, New York, 1966
- [3] I. S. Romanovskii, *Free subgroups in finitely presented groups*, Algebra Logic **16** (1978), 62–68
- [4] L. M. Shneerson, *On free subsemigroups in finitely presented semigroups*, Siberian Math. J. **15** (1974), 325–328
- [5] L. M. Shneerson, *On free subalgebras in finitely presented algebras* [in Russian], IV USSR Symposium on the theory of rings, algebras and modules, Kishinev, 1980, 117–118
- [6] O. G. Kharlampovich, *Lyndon's condition for solvable Lie algebras* [in Russian], Izv. Vyssh. Uchebn. Zaved. Mat. **28** (1984), 50–59